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A ONE-DIMENSIONAL MODEL FOR SEASONAL VARIATION OF TEMPERATURE DISTRIBUTION IN STRATIFIED LAKES

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Abstract—A one-dimensional, two-layer model describing the thermal structure in large bodies of stratified water is developed. The model accounts for the nonlinear interaction between the wind induced turbulence and the buoyancy gradients produced by surface heating, the effects of the attenuation of the solar radiation in the body of water. The model is applied to predict the seasonal variation of the depth of the thermocline and the vertical temperature distribution during the stratification period of Cayuga Lake, New York. The results agreed fairly well with the field data.

NOMENCLATURE

C_p, specific heat;
g, acceleration due to gravity;
H, depth of lake;
$h(t)$, depth of the upper layer;
$\bar{I}$, average value of the solar radiation intensity;
ΔI, half of the annual variation of solar radiation intensity;
t, time;
$T(z,t)$, temperature of the lower layer;
T_e, equilibrium temperature defined by equation (8);
$\bar{T}_e$, an average value of the equilibrium temperature;
ΔT_e, half of the annual variation of the equilibrium temperature;
T_H, temperature at the bottom of the lake;
$T_s(t)$, temperature of the upper layer;
q, turbulent heat flux;
q^r, radiative heat flux;
w^*, $= \sqrt{(\tau_s/\rho)}$, friction velocity;
z, vertical distance measured downward from the surface.

Greek symbols

α, $\int_0^1 \theta(\eta)\,d\eta = 0.75$;
δ, coefficient of volumetric expansion of water;
κ, von Kármán constant, ≈ 0.4;
β, extinction coefficient;
ϕ, ϕ', phase angles associated with the equilibrium temperature and the solar radiation intensity respectively;
γ, $\int_0^1 \eta\theta(\eta)\,d\eta = 0.45$;
η, dimensionless variable defined by equation (3);
ω, single scattering albedo;
ρ, density of water;
σ, Stefan–Boltzmann constant;
τ, $= \beta z$, optical variable;
τ_0, $= \beta H$, optical depth of the lake;
τ_s, surface shear stress induced by wind;
θ, dimensionless temperature defined by equation (2).

INTRODUCTION

LARGE bodies of water such as lakes provide a convenient source of cooling water supply to electric generating power plants. The cold water available at depth in lakes is used in the stream condensers and then returned back to the lake. A knowledge of the temperature structure within a large body of water and the changes that take place in it are essential before the perturbing effects of the added heat load on the lake temperature can be assessed. Quantitative features of the stratification cycle of lakes have been described in the literature [1, 2] and the seasonal stratification cycle of temperature has been studied by various investigators [3, 7]. In the present study a one-dimensional, two-layer model is developed for the prediction of the seasonal variation of the vertical temperature distribution in stratified lakes. The model allows for the surface heat exchange, the interaction between the wind induced turbulence and buoyancy gradients, and includes the solar heating effects as a bulk process by considering the absorption and scattering of the solar radiation with depth in the body of water.

ANALYSIS

The one-dimensional, time dependent energy equation for an incompressible fluid under the assumption of horizontal homogeneity is taken as

$$\rho C_p \frac{\partial T(z,t)}{\partial t} = -\frac{\partial}{\partial z}(q+q^r) \quad \text{in} \quad 0 \leqslant z \leqslant H \tag{1}$$

where T is the temperature, t is the time, q and q^r are the turbulent and radiative heat fluxes respectively, z is the vertical coordinate in the body of water measured from the surface, ρ is the density, C_p is the specific heat and H is the depth of the lake. A two-layer model is now considered for the vertical temperature profile in the lake:

(i) A well mixed upper layer in the region $0 \leqslant z \leqslant h(t)$ where the vertical temperature distribution is considered uniform and taken as $T_s(t)$, and

(ii) A lower layer in the region $h(t) \leqslant z \leqslant H$ where the temperature varies from $T_s(t)$ at $z = h(t)$ to a constant value T_H at the bottom of the lake $z = H$. It is further assumed that the lake is sufficiently deep, hence the heat losses at the bottom of the lake are negligible. For convenience in the analysis a dimensionless temperature θ and a dimensionless coordinate η are defined for the lower layer as:

$$\theta(\eta) = \frac{T_s(t) - T(z,t)}{T_s(t) - T_H}, \quad \text{in} \quad h(t) \leqslant z \leqslant H \tag{2}$$

and

$$\eta = \frac{z - h(t)}{H - h(t)}, \quad \text{in} \quad h(t) \leqslant z \leqslant H. \tag{3}$$

If $\theta(\eta)$ is represented by a third-degree polynomial in the form

$$\theta(\eta) = A_0 + A_1\eta + A_2\eta^2 + A_3\eta^3 \quad \text{in} \quad 0 \leqslant \eta \leqslant 1 \tag{4a}$$

with the boundary conditions

$$\theta(\eta) = 0 \quad \text{at} \quad \eta = 0, \tag{4b}$$

$$\theta(\eta) = 1, \quad \frac{d\theta}{d\eta} = 0 \quad \text{and} \quad \frac{d^2\theta}{d\eta^2} = 0 \quad \text{at} \quad \eta = 1, \tag{4c}$$

one finds that the temperature profile $\theta(\eta)$ in the lower layer is represented by

$$\theta(\eta) = 3\eta - 3\eta^2 + \eta^3, \qquad 0 \leqslant \eta \leqslant 1 \tag{5}$$

Clearly, if $T_s(t)$ and $h(t)$ are known, the corresponding η and $\theta(\eta)$ are determined from equations (3) and (5) respectively, and the temperature distribution $T(z, t)$ in the lower layer from equation (2) as

$$T(z,t) = T_s(t) - [T_s(t) - T_H]\theta(\eta), \quad h(t) \leqslant z \leqslant H. \tag{6}$$

Two equations that are needed for the determination of $T_s(t)$ and $h(t)$ are now derived from the energy equation (1) as described below.

The energy equation (1) is integrated with respect to z over the entire depth of the lake from $z = 0$ to $z = H$ to yield

$$\rho C_p \left[h\frac{dT_s}{dt} + \frac{\partial}{\partial t}\int_{h(t)}^{H} T(z,t)\,dz + T_s\frac{dh}{dt} \right] = q_s \tag{7}$$

where the radiative heat flux at the bottom of the lake is assumed to be zero and q_s represents the sum of the turbulent and radiative heat fluxes at the surface. Now, q_s is represented in the form [8]

$$q_s = K'(T_e - T_s) \tag{8}$$

where K' is the heat exchange coefficient at the surface and T_e is an equilibrium temperature, and the integral term in equation (7) is evaluated by utilizing equations (2), (3) and (5) as

$$\int_{h(t)}^{H} T(z,t)\,dz = (H-h)T_s - \alpha(T_s - T_H)(H-h) \tag{9}$$

with

$$\alpha \equiv \int_0^1 \theta(\eta)\,d\eta = 0{\cdot}75. \tag{10}$$

The substitution of equations (8)–(10) into equation (7) gives the first of these two equations as

$$(H - \alpha H + \alpha h)\frac{dT_s}{dt} + \alpha(T_s - T_H)\frac{dh}{dt} = \frac{K'}{\rho C_p}(T_e - T_s) \tag{11a}$$

where the annual variation of the equilibrium temperature, T_e, is represented by [9]

$$T_e = \bar{T}_e + \Delta T_e \sin\left(\frac{2\pi}{365}t + \phi\right). \tag{11b}$$

Here $\bar{T}_e$ is an average value, ΔT_e is half of the annual variation and ϕ depends upon the conditions from which the computations begin.

A second equation is obtained by taking the first moment of equation (1); that is, equation (1) is multiplied by z and integrated from $z = 0$ to $z = H$ to yield

$$\tfrac{1}{2}h^2\frac{dT_s}{dt} + \frac{d}{dt}\int_h^H zT(z,t)\,dz + hT_s\frac{dh}{dt} = \int_0^H \frac{q}{\rho C_p}\,dz + \int_0^H \frac{q^r}{\rho C_p}\,dz \tag{12}$$

where q and q^r are the turbulent and radiative heat fluxes respectively. Equations (2), (3) and (5) are introduced into equation (12), the integration involving the temperature is performed by noting that the temperature is independent of z and equal to $T_s(t)$ in the upper layer, $0 \leqslant z \leqslant h(t)$; after some manipulation one obtains

$$\begin{aligned}&[(\alpha-\gamma)h^2 + H(2\gamma-\alpha)h + H^2(\tfrac{1}{2}-\gamma)]\frac{dT_s}{dt} \\ &\quad + [2(\alpha-\gamma)h + (2\gamma-\alpha)H][T_s - T_H]\frac{dh}{dt} \\ &= \int_0^H \frac{q}{\rho C_p}\,dz + \int_0^H \frac{q^r}{\rho C_p}\,dz\end{aligned} \tag{13}$$

with

$$\gamma \equiv \int_0^1 \eta\theta\,d\eta = 0{\cdot}45. \tag{14}$$

The turbulent heat transfer term on the RHS of equation (13) can be related [4, 9, 10] to the wind stress acting on the water surface by making use of the fact that the thermal stratification in a lake acts as a barrier to mixing, while the wind stress creates turbulence that acts against the buoyancy gradient. Therefore, a mechanical energy balance in the water relates the kinetic energy input from the wind directly to the transformation of the potential energy into kinetic energy by convection within the layer if the turbulent energy dissipation due to viscosity is neglected; the kinetic energy input into the water is then related to the wind stress at the water surface [4, 10]. With an analysis based on these considerations it can be shown that the integral term involving the turbulent heat flux in equation (13) is related to the wind stress τ_s at the surface by [4, 9–12]

$$\int_0^H \frac{q}{\rho C_p}\,dz \cong \frac{w^{*3}}{\delta g \kappa} \qquad (15a)$$

where

$$w^* = \sqrt{\left(\frac{\tau_s}{\rho}\right)} = \text{friction velocity}. \qquad (15b)$$

The determination of the radiative heat flux, q^r, however, requires the solution of the equation of radiative transfer over the entire depth of the lake. The radiation part of the problem to account for the bulk heating of the water due to the penetration of the solar radiation is treated by considering a plane-parallel, absorbing, emitting, isotropically scattering gray medium with azimuthal symmetry. The P_1-approximation of the spherical harmonics method is used to solve the radiation problem. In this method the equation of radiative transfer takes the form [13]

$$\frac{d^2G(\tau)}{d\tau^2} - K^2(\tau) = -4K^2\sigma T(\tau, t) \quad \text{in } 0 \leqslant \tau \leqslant \tau_0 \qquad (16)$$

where

$$K^2 = 3(1-\omega)$$

$T(\tau, t)$ = temperature distribution in the lake.

Once the function $G(\tau)$ is known from the solution of equation (16) subject to appropriate boundary conditions, the net radiative heat flux $q^r(\tau)$ is determined from

$$q^r(\tau) = -\tfrac{1}{3}\frac{dG(\tau)}{d\tau}. \qquad (17)$$

We note that equation (16) is coupled to the energy equation because it contains the unknown temperature distribution function $T(\tau, t)$. For most lakes the source term on the RHS of equation (16) is very small compared to the solar radiation energy incident on the lake surface. Then the coupling is removed and equation (16) is simplified as:

$$\frac{dG(\tau)}{d\tau^2} - K^2G(\tau) = 0 \quad \text{in} \quad 0 \leqslant \tau \leqslant \tau_0. \qquad (18)$$

The boundary conditions for this equation are established by assuming that the solar radiation incident on the lake surface is specified and that no radiation is coming from the bottom of the lake. With this consideration the boundary conditions for equation (18) are taken in the Marshak boundary condition approximation as [13].

$$G(\tau) - \tfrac{2}{3}\frac{dG(\tau)}{d\tau} = 4\pi[\bar{I} + \Delta I \sin(\Omega t + \phi')], \ \tau = 0 \qquad (19a)$$

$$G(\tau) + \tfrac{2}{3}\frac{dG(\tau)}{d\tau} = 0, \qquad \tau = \tau_0. \qquad (19b)$$

The boundary condition (19a) assumes that the annual variation of the intensity I of the solar radiation incident on the water surface is specified as

$$\bar{I} + \Delta I \sin(\Omega t + \phi')$$

where I is an average value and ΔI is half of the annual variation of the solar radiation intensity, $\Omega = 2\pi/365$ day^{-1} and the value of ϕ' depends on the conditions at the start of computations.

The solution of equation (18) subject to the boundary conditions (19) is straightforward. Knowing $G(\tau)$, the net radiative heat flux $q^r(\tau)$ is obtained from equation (17) as

$$q^r(\tau) = \frac{\tfrac{4}{3}\pi K\left[\bar{I} + \Delta I \sin\left(\frac{2\pi}{365}t + \phi'\right)\right]}{\tfrac{2}{3}K\cosh(K\tau_0) + (1 + \tfrac{4}{9}K^2)\sinh(K\tau_0)} \cdot \{[\cosh(K\tau_0) + \tfrac{2}{3}K\sinh(K\tau_0)]\cosh(K\tau) - [\sinh(K\tau_0) + \tfrac{2}{3}K\cosh(K\tau_0)]\sinh(K\tau)\}. \qquad (20)$$

Noting that $\tau = \beta z$ and $\tau_0 = \beta H$, the integral term $\int_0^H q^r\,dz$ in equation (13) becomes

$$\int_0^H q^r\,dz = \frac{\pi\left[\bar{I} + \Delta I \sin\left(\frac{2\pi}{365}t + \phi'\right)\right]\cdot\left\{\tanh(K\beta H) + \tfrac{2}{3}K\left[1 - \frac{1}{\cosh(K\beta H)}\right]\right\}}{\beta K + (\tfrac{3}{4} + \tfrac{1}{3}K^2)\beta\tanh(K\beta H)}. \qquad (21)$$

Introducing equations (15) and (21) into equation (13) the desired second equation becomes

$$[(\alpha-\gamma)h^2 + H(2\gamma-\alpha)h + H^2(\tfrac{1}{2}-\gamma)]\frac{dT_s}{dt} + [2(\alpha-\gamma)h + (2\gamma-\alpha)H]\cdot(T_s - T_H)\frac{dh}{dt}$$
$$= \frac{w^{*3}}{\delta g \kappa} + \frac{\pi\left[\bar{I} + \Delta I \sin\left(\frac{2\pi}{365}t + \phi'\right)\right]\cdot\left\{\tanh(K\beta H) + \tfrac{2}{3}K\left[1 - \frac{1}{\cosh(K\beta H)}\right]\right\}}{\rho C_p[\beta K + (\tfrac{3}{4} + \tfrac{1}{3}K^2)\beta\tanh(K\beta H)]} \qquad (22)$$

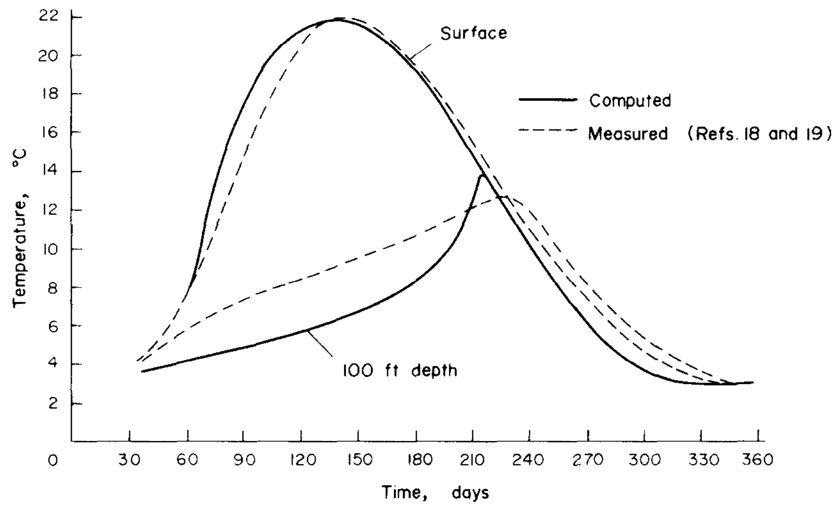

FIG. 1. Comparison of the computational and observed stratification cycle of Cayuga Lake, New York.

To summarize, equations (11) and (22) provide two coupled, first-order nonlinear ordinary differential equations for the determination of the temperature $T_s(t)$ in the upper layer and the depth $h(t)$ of the thermocline. Then, the temperature distribution in the lower layer is determined by equations (3), (5) and (6).

RESULTS

Equations (11) and (22) were solved numerically by using a Runge–Kutta method. The computations were performed for the actual conditions that correspond to Cayuga Lake, New York [14] with the input parameters taken as [14, 15]

$$T_e = 11 + 16\sin\left(\frac{2\pi}{365}t - 0{\cdot}531\right), \ ^\circ\mathrm{C}$$

$$K' = 180\,\mathrm{Btu/ft^2\,day\,^\circ C}$$

$$H = 200\,\mathrm{ft}$$

$$I = 1955 + 1120\sin\left(\frac{2\pi}{365}t - 0{\cdot}049\right), \ \mathrm{Btu/ft^2\,day}.$$

The semi-empirical relation between the wind stress τ_s at the surface and the wind speed given by Munk and Anderson [16] is used to evaluate the friction velocity w^*. The minimum temperature during spring homothermy was assumed to be 2·9°C. The absorption and scattering coefficients for water and the particles in suspension were assumed to be $0{\cdot}31\,\mathrm{ft}^{-1}$ and $0{\cdot}63\,\mathrm{ft}^{-1}$ respectively; these values were estimated from recent data given in reference [17]. The calculations started from the minimum surface temperature of the lake that corresponded to the homothermal state. To avoid the computational difficulty at time $t = 0$, calculations were started with some specific values of $T_s(t)$ and $h(t)$ slightly away from the origin. Figure 1 shows a comparison of the computed and observed [18, 19] values of temperature cycle for the Cayuga Lake, New York. Figure 2 is a comparison of the computed and observed [18] depth of the thermocline as a function of time. The agreement between the observed and computed values is fairly good.

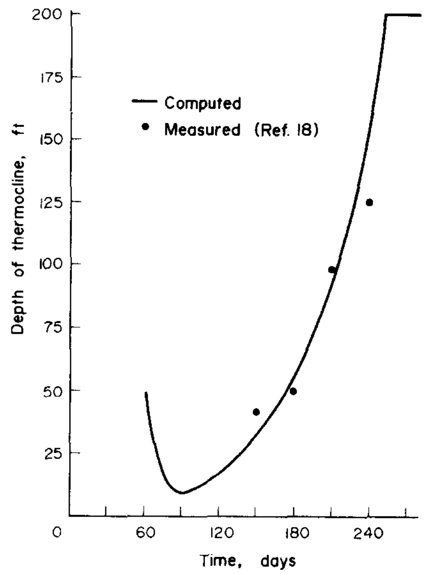

FIG. 2. Comparison of calculated and observed thermocline depths for Cayuga Lake, New York.

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MODELE UNIDIMENSIONNEL POUR LES VARIATIONS SAISONNIERES DES DISTRIBUTIONS DE TEMPERATURE DANS LES LACS STRATIFIES

Résumé—Un modèle unidimensionnel à deux couches est développé afin de décrire la structure thermique des grandes étendues d'eau stratifiées. Le modèle tient compte de l'intéraction non-linéaire entre la turbulence causée par le vent, les gradients des forces de gravité produits par l'échauffement de la surface et des effets de l'atténuation du rayonnement solaire au sein de l'eau. Le modèle est appliqué à la prévision des variations saisonnières de la profondeur des thermoclines et la destribution verticale des températures pendant la période de stratification du lac Cayuga (New-York). Les résultats sont en assez bon accord avec l'ensemble des mesures.

EIN EINDIMENSIONALES MODELL FÜR DIE SAISONBEDINGTEN VARIATIONEN DER TEMPERATURVERTEILUNG IN GESCHICHTETEN GEWÄSSERN

Zusammenfassung—Es wird ein eindimensionales Zweilagenmodell entwickelt, das die thermische Struktur in großen, geschichteten Wassermengen beschreibt. Das Modell berücksichtigt den nichtlinearen Zusammenhang zwischen den windinduzierten Turbulenzen und den Auftriebsgradienten aufgrund der Oberflächenerwärmung wie den Einfluß der Absorption der Solarstrahlung im Wasserkörper. Es dient zur Vorhersage der saisonbedingten Schwankungen der Tiefenlage der Thermokline und der vertikalen Temperaturverteilung während der Schichtungszeit des Cayuga-Sees, New York. Die Ergebnisse stimmen relativ gut mit den Felddaten überein.

ОДНОМЕРНАЯ МОДЕЛЬ СЕЗОННОГО ИЗМЕНЕНИЯ ТЕМПЕРАТУРЫ В СЛОЯХ ОЗЕР

Аннотация — Создана одномерная двухслойная модель для описания термоструктуры в больших массах стратифицированной воды. В модели учитывается нелинейное взаимодействие между вызванной ветром турбулентностью и перепадами в подъемной силе, вызванными нагреванием поверхности, эффектами затухания солнечной радиации в массе воды. Модель используется для расчета сезонного изменения глубины термоклимата и вертикального распределения температуры в период стратификации на озере Кайюга шт. Нью-Йорк. Результаты расчетов довольно хорошо согласуются с экспериментальными данными.